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On the Bourque–Ligh Conjecture of Least Common Multiple Matrices

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Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The matrix $[S]_n$ having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix on S. A set S is gcd-closed if $(x_i, x_j) \in S$ for $1 \leq i, j \leq n$. Bourque and Ligh conjectured that the LCM matrix $[S]_n$, defined on a gcd-closed set S, is nonsingular. In this paper we prove that the conjecture is true for $n \leq 7$ and is not true for $n \geq 8$. So the conjecture is solved completely.

I. INTRODUCTION

Smith [8] showed that the determinant of the $n \times n$ matrix $[(i, j)]_n$ which has the greatest common divisor (i, j) of i and j as its i, j-entry is the product $\phi(1)\phi(2)\cdots\ \phi(n)$, where ϕ is Euler's totient function. He also proved that if f is an arithmetical function and $[f(i, j)]n$ is the $n \times n$ matrix having f evaluated at the greatest common divisor (i, j) of i and j as its i, j-entry, then $\det[f(i, j)]_n = (f * \mu)(1)(f * \mu)(2) \cdots (f * \mu)(n)$, where μ is the Möbius function and $f * \mu$ is the Dirichlet convolution of f and μ. In 1972, Apostol [1] extended Smith's result. In 1988, McCarthy [7] generalized Smith's and Apostol's results to the class of even functions (mod r). In 1993, Bourque and Ligh [4] extended the results of Smith, of Apostol, and of McCarthy.

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The matrix $(S)_n$, having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry, is called the greatest common divisor (GCD) matrix on S. The matrix $[S]_n$ having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix on S. The set

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S is called factor-closed if it contains every divisor of x for any $x \in S$. Clearly, for any positive integer n, the set $\{1, 2, \dots, n\}$ is factor-closed. A set S is called gcd-closed, if $(x_i, x_j) \in S$ for $1 \le i, j \le n$. Obviously, a factor-closed set is gcd-closed but not conversely. For example, $S = \{1, 2, 3, 8, 10\}$ is gcd-closed but not factor-closed.

In 1989, Beslin and Ligh [2] investigated the structure of the GCD matrix $(S)_n$ on a set S of distinct positive integers. In 1992, Bourque and Ligh [3] obtained formulas for the inverses of the GCD matrix $(S)_n$ and the LCM matrix $[S]_n$ on a factor-closed set S. In the meantime they obtained a formula for the inverses of the GCD matrix $(S)_n$ and a formula for the determinant of the LCM matrix $[S]_n$ on a gcd-closed set as

$$\det[S]_n = \prod_{k=1}^{n} x_k^2 \alpha_k, \tag{1}$$

where

$$\alpha_k = \alpha_k(x_1, \dots, x_k) = \sum_{\substack{d \mid x_k, \\ d \nmid x_t, \\ x_t < x_k.}} g(d), \tag{2}$$

with the arithmetical function g defined by $g(m) = \frac{1}{m} \cdot \sum_{d \mid m} d \cdot \mu(d)$, and the function μ is the Möbius function. In [3], Bourque and Ligh raised the following

Conjecture. The LCM matrix $[S]_n$ on a gcd-closed set S is nonsingular.

Recently, Haukkanen, Wang, and Sillanpää [5] gave a counterexample for $n = 9$. In [6] we show that the conjecture is true for $n \le 5$.

In this paper we introduce a new method to reduce, greatly, the formula for α_k. We prove that the conjecture is true for $n \le 7$ and is not true for $n \ge 8$. So the conjecture is solved completely.

II. THE MAIN LEMMAS

Throughout this section, let the set $S = \{x_1, \dots, x_n\}$ of n distinct positive integers be gcd-closed and $1 \le x_1 < \cdots < x_n$. Let $|A|$ denote the cardinality of any finite set A.

LEMMA 1 ([6]). *For* $1 \le k \le n$, *we have*

$$\alpha_k = \beta_k(x_1, \dots, x_k), \tag{3}$$

where the function β_k *on a set* $Z = \{z_1, \dots, z_k\}$ *of* k *distinct positive integers is defined as*

$$\beta_k(z_1, \dots, z_k) = \frac{1}{z_k} + \sum_{r=1}^{k-1} (-1)^r \cdot \sum_{1 \le i_1 < \cdots < i_r \le k-1} \frac{1}{(z_{i_1}, \dots, z_{i_r}, z_k)}, \tag{4}$$

where $(z_{i_1}, \dots, z_{i_r}, z_k)$ *denotes the greatest common divisor of* $z_{i_1}, \dots, z_{i_r}, z_k$.

Therefore we can compute α_k by either (2) or (3). It is obvious that formula (3) is easier to use than (2). Now we give further reduction for the formula of α_k.

LEMMA 2. *Let* $T_k = \{x \in S \mid x < x_k$ *and* $x \nmid x_k\}$, $1 \le k \le n$. *Let* $\{x_1, \dots, x_k\} \setminus T_k = \{x_{k,1}, x_{k,2}, \dots, x_{k,r_k}\}$, *where* $x_1 = x_{k,1} < x_{k,2} < \cdots < x_{k,r_k} = x_k$, $2 \le r_k \le k$. *Then*

$$\alpha_k = \beta_{r_k}(x_{k,1}, x_{k,2}, \dots, x_{k,r_k}). \tag{5}$$

Proof. If $T_k = \phi$, then it follows from Lemma 1 that (5) holds. If $T_k \ne \phi$. Let $x_t \in T_k$. Then $x_t < x_k$. Let $\{x_1, \dots, x_k\} \setminus \{x_t\} = \{x'_1, \dots, x'_{k-2}, x_k\}$, where $x'_1 < \cdots < x'_{k-2} < x_k$. Since

$$\beta_k(x_1, \dots, x_k) = \frac{1}{x_k} + \Delta'_t + \Delta_t, \tag{6}$$

where

$$\Delta'_t = \sum_{r=1}^{k-2} (-1)^r \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t}} \frac{1}{(x_{i_1}, \dots, x_{i_r}, x_k)}, \tag{7}$$

$$\Delta_t = \sum_{r=0}^{k-2} (-1)^{r+1} \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t}} \frac{1}{(x_{i_1}, \dots, x_{i_r}, x_t, x_k)}, \tag{8}$$

by the definition of β_{k-1} we have

$$\frac{1}{x_k} + \Delta'_t = \beta_{k-1}(x'_1, \dots, x'_{k-2}, x_k).$$

Thus,

$$\beta_k(x_1, \dots, x_k) = \beta_{k-1}(x'_1, \dots, x'_{k-2}, x_k) + \Delta_t. \tag{9}$$

Since S is gcd-closed, $(x_k, x_t) \in S$. Let $x_l = (x_k, x_t)$. Assume that $x_l = x_t$. Then $x_t \mid x_k$. Thus $x_t \notin T_k$. It is a contradiction. So $x_l < x_t$, i.e., $l < t$. Therefore,

$$\begin{aligned}
\Delta_t &= \sum_{r=0}^{k-3} (-1)^{r+1} \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t, \\ i_j \ne l.}} \frac{1}{(x_{i_1}, \ldots, x_{i_r}, x_t, x_k)} + \sum_{r=0}^{k-3} (-1)^{r+2} \\
&\quad \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t, \\ i_j \ne l.}} \frac{1}{(x_{i_1}, \ldots, x_{i_r}, x_l, x_t, x_k)} \\
&= \sum_{r=0}^{k-3} \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t, \\ i_j \ne l.}} \left(\frac{(-1)^{r+1}}{(x_{i_1}, \ldots, x_{i_r}, x_t, x_k)} \right. \\
&\qquad \left. + \frac{(-1)^{r+2}}{(x_{i_1}, \ldots, x_{i_r}, x_l, x_t, x_k)} \right) \\
&= \sum_{r=0}^{k-3} \sum_{\substack{1 \le i_1 < \cdots < i_r \le k-1 \\ i_j \ne t, \\ i_j \ne l.}} \left(\frac{(-1)^{r+1}}{(x_{i_1}, \ldots, x_{i_r}, x_l)} + \frac{(-1)^{r+2}}{(x_{i_1}, \ldots, x_{i_r}, x_l)} \right) \\
&= 0.
\end{aligned}$$

By (9) we have that

$$\beta_k(x_1, \ldots, x_k) = \beta_{k-1}(x'_1, \ldots, x'_{k-2}, x_k), \tag{10}$$

where $\{x'_1, \ldots, x'_{k-2}, x_k\} = \{x_1, \ldots, x_k\} \setminus \{x_t\}$.

If $T_k \setminus \{x_t\} = \phi$, i.e., $T_k = \{x_t\}$, then it follows that (5) holds from (10) and (3). If $T_k \setminus \{x_t\} \ne \phi$, then the above reduction can continue. Note that T_k is finite. So we can repeat $|T_k|$ times the above reduction. This completes the proof. ∎

Definition. Let T be a set of distinct positive integers. For any $a, b \in T$ and $a < b$, we say that a is the greatest-type divisor of b in T, if $a \mid b$ and it can be deduced that $c = a$ from $a \mid c$, $c \mid b$, $c < b$, and $c \in T$.

For example, $T = \{1, 2, 3, 4, 5, 6, 8\}$. Then 2 is the greatest-type divisor of 4 and 6 in T, but 2 is not the greatest-type divisor of 8 in T.

LEMMA 3. *Let $R_k = \{y_{k,1}, \dots, y_{k,l_k}\}$ be the set of the greatest-type divisors of x_k in S, where $y_{k,1} < \cdots < y_{k,l_k}$, $l_1 = 0$, $l_2 = l_3 = 1$, and $1 \le l_k \le k - 2$ for $k \ge 4$. Then*

$$\alpha_k = \beta_{l_k+1}(y_{k,1}, \dots, y_{k,l_k}, x_k). \tag{11}$$

Proof. If $k = 1$, then $R_1 = \phi$. So $\alpha_1 = \beta_1(x_1)$. Thus (11) holds. If $k = 2$, then $R_1 = \{x_1\}$. By (3) we have that $\alpha_2 = \beta_2(x_1, x_2)$. Then (11) holds. In the following we let $k \ge 3$. Let $T_k = \{x \in S \mid x < x_k,\ x \nmid x_k\}$, $S'_k = \{x_1, \dots, x_k\} \setminus T_k$. Write $S'_k = \{x'_1, \dots, x'_h\}$, where $x_1 = x'_1 < \cdots < x'_h = x_k$ and $2 \le h = k - |T_k| \le k$. Then $R_k \subset S'_k$ and R_k is equal to the set of the greatest-type divisors of x_k in S'_k. If $h = 2$, then $S'_k = \{x_1, x_k\}$ and $T_k = \{x_2, \dots, x_{k-1}\}$. Thus $R_k = \{x_1\}$. By (5) we have that $\alpha_k = \beta_2(x_1, x_k)$ so (11) holds. In the following we let $h \ge 3$. We claim that $\{x'_1, \dots, x'_{h-1}\} \setminus R_k \ne \phi$. Otherwise, $R_k = \{x'_1, \dots, x'_{h-1}\}$. Since $x_1 | x_i$ $(i > 1)$ (since S is gcd-closed), one has $x'_1 | x_i$ $(i > 1)$. Note that $x'_{h-1} \in \{x_2, \dots, x_{k-1}\}$. So $x'_1 | x'_{h-1}$. Then $x'_1 \notin R_k$. It is a contradiction. Thus the claim is true.

Let $x'_t \in \{x'_1, \dots, x'_{h-1}\} \setminus R_k$. Then $x'_t \notin R_k$. So there exists $x \in \{x'_1, \dots, x'_{h-1}\} \setminus \{x'_t\}$, such that $x'_t | x$. Let $R = \{z_1, \dots, z_p\}$ $(p \ge 1)$ be the set of such element x. Namely, $\{z_1, \dots, z_p\} = \{x | x \in \{x'_1, \dots, x'_{h-1}\} \setminus \{x'_t\}$, and $x'_t | x\}$. Then $x'_t | z_i$ $(1 \le i \le p)$. Let $\{w_1, \dots, w_q\} = \{x'_1, \dots, x'_{h-1}\} \setminus (R \cup \{x'_t\})$, where $q = h - p - 2$. By Lemma 2 we have that

$$\alpha_k = \beta_h(x'_1, \dots, x'_h).$$

Note that $x'_h = x_k$. In a way similar to that in (6), we have that

$$\alpha_k = \frac{1}{x_k} + \overline{\Delta'_t} + \overline{\Delta_t}, \tag{12}$$

where

$$\overline{\Delta'_t} = \sum_{r=1}^{h-2} (-1)^r \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le h-1 \\ i_j \ne t}} \frac{1}{(x'_{i_1}, \dots, x'_{i_r}, x_k)}, \tag{13}$$

$$\overline{\Delta_t} = \sum_{r=0}^{h-2} (-1)^{r+1} \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le h-1 \\ i_j \ne t}} \frac{1}{(x'_{i_1}, \dots, x'_{i_r}, x'_t, x_k)}. \tag{14}$$

For $\overline{\Delta_t}$ we have

$$\overline{\Delta_t} = \sum_{r=0}^{q} (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le q} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)}$$
$$+ \sum_{r=0}^{q} \sum_{e=1}^{p} (-1)^{r+e+1} \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le q, \\ 1 \le j_1 < \cdots < j_e \le p.}} \frac{1}{(w_{i_1}, \dots, w_{i_r}, z_{j_1}, \dots, z_{j_e}, x'_t, x_k)}. \tag{15}$$

Since $x'_t | z_i$ $(1 \le i \le p)$, we have that $(z_{j_1}, \dots, z_{j_e}, x'_t) = x'_t$ for any $1 \le j_1 < \cdots < j_e \le p$. So we have that $(w_{i_1}, \dots, w_{i_r}, x'_t, x_k) = (w_{i_1}, \dots, w_{i_r}, z_{j_1}, \dots, z_{j_e}, x'_t, x_k)$ for any $1 \le j_1 < \cdots < j_e \le p$ and $1 \le i_1 < \cdots < i_r \le q$. Then, by (15), we have that

$$\begin{aligned}
\overline{\Delta_t} &= \sum_{r=0}^{q} (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le q} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)} \\
&\quad + \sum_{r=0}^{q} \sum_{e=1}^{p} (-1)^{r+e+1} \cdot \sum_{\substack{1 \le i_1 < \cdots < i_r \le q, \\ 1 \le j_1 < \cdots < j_e \le p.}} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)} \\
&= \sum_{r=0}^{q} (-1)^{r+1} \cdot \sum_{1 \le i_1 < \cdots < i_r \le q} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)} \\
&\quad \times \left(1 + \sum_{e=1}^{p} (-1)^{e} \cdot \sum_{1 \le j_1 < \cdots < j_e \le p} 1\right) \\
&= \sum_{r=0}^{q} (-1)^{r+1} \cdot \sum_{1 \le i_1 < \cdots < i_r \le q} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)} \\
&\quad \times \left(1 + \sum_{e=1}^{p} (-1)^{e} \cdot \binom{p}{e}\right) \\
&= \sum_{r=0}^{q} (-1)^{r+1} \cdot \sum_{1 \le i_1 < \cdots < i_r \le q} \frac{1}{(w_{i_1}, \dots, w_{i_r}, x'_t, x_k)} \cdot (1-1)^{p} \\
&= 0.
\end{aligned}$$

By (12), and the definition of β_{h-1}, we have that

$$\alpha_k = \beta_{h-1}(x'_1, \dots, x'_{t-1}, x'_{t+1}, \dots, x'_h). \tag{16}$$

If $\{x'_1, \dots, x'_{h-1}\} \setminus (R_k \cup \{x'_t\}) = \phi$, then $R_k = \{x'_1, \dots, x'_{h-1}\} \setminus \{x'_t\}$. So (11) holds by (16). If $\{x'_1, \dots, x'_{h-1}\} \setminus (R_k \cup \{x'_t\}) \ne \phi$, then the above reduction can continue. Let $L_k = \{x'_1, \dots, x'_{h-1}\} \setminus R_k$. Then L_k is finite. So (11) holds by using $|L_k|$ times the above reduction. This completes the proof. ∎

In the following lemmas let $R_k = \{y_1, y_2, \dots, y_m\}$ be the set of the greatest-type divisors of x_k $(1 \le k \le n)$ in S, where $y_1 < y_2 < \cdots < y_m$. Then $R_1 = \phi$ and $R_k \ne \phi$ for $k \ge 2$. If $m \ge 2$, we suppose that $G = (y_1, \dots, y_m)$ and $y_i = G \cdot y'_i$ $(1 \le i \le m)$. Then $1 < y'_1 < y'_2 < \cdots < y'_m$ and $(y'_1, y'_2, \dots, y'_m) = 1$. Define $M^{(m)} = \bigcup_{r=2}^{m} M_r^{(m)}$, where $M_r^{(m)} = \{(y_{i_1}, \dots, y_{i_r}) | 1 \le i_1 < \cdots < i_r \le m\}$ $(2 \le r \le m)$, and the $(y_{i_1}, \dots, y_{i_r})$ denotes the greatest common divisor of $y_{i_1}, \dots, y_{i_r}$. So $G \in M^{(m)}$ and $|M^{(m)}| \ge 1$. Thus $G | x$ for any $x \in M^{(m)}$.

LEMMA 4. *Let positive integer $m \le 2$. Then $\beta_{m+1}(y_1, \dots, y_m, x_k) \ne 0$.*

Proof. If $m = 1$, then $\beta_2(y_1, x_k) = (1/x_k) - (1/y_1)$ by (4). Since $y_1 < x_k$, $\beta_2(y_1, x_k) < 0$. If $m = 2$, noting that $(y_1, y_2) = G$, then by (4) we have that

$$\begin{aligned}\beta_3(y_1, y_2, x_k) &= \frac{1}{x_k} - \frac{1}{y_1} - \frac{1}{y_2} + \frac{1}{G} \\ &= \frac{1}{x_k} + \frac{1}{G}\left(1 - \frac{1}{y'_1} - \frac{1}{y'_2}\right). \end{aligned} \tag{17}$$

Since $y'_1 \ge 2$ and $y'_2 \ge 3$,

$$1 - \frac{1}{y'_1} - \frac{1}{y'_2} \ge 1 - \frac{1}{2} - \frac{1}{3} > 0.$$

Thus by (17) we have that $\beta_3(y_1, y_2, x_k) > 0$. The proof is complete. ∎

LEMMA 5. *Let $m \ge 3$. If $|M^{(m)}| = 1$, then $\beta_{m+1}(y_1, \dots, y_m, x_k) \ne 0$.*

Proof. Since $G \in M^{(m)}$ and $|M^{(m)}| = 1$, $M^{(m)} = \{G\}$. Then we have that $(y_{i_1}, \dots, y_{i_r}) = G$ for any $1 \le i_1 < \cdots < i_r \le m$ $(2 \le r \le m)$. It implies that $(y'_{i_1}, \dots, y'_{i_r}) = 1$. By (4) we have that

$$\begin{aligned}\beta_{m+1}(y_1, \dots, y_m, x_k) &= \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \sum_{r=2}^{m} (-1)^r \cdot \sum_{1 \le i_1 < \cdots < i_r \le m} \frac{1}{G} \\ &= \frac{1}{x_k} - \sum_{i=1}^{m} \frac{1}{y_i} + \frac{1}{G} \cdot \sum_{r=2}^{m} (-1)^r \cdot \binom{m}{r} \\ &= \frac{1}{x_k} + \frac{1}{G}\left(m - 1 - \sum_{i=1}^{m} \frac{1}{y'_i}\right). \end{aligned} \tag{18}$$

Since $y_2' > y_1' \geq 2$,

$$1 - \frac{1}{y_1'} - \frac{1}{y_2'} > 0.$$

For $i = 3, \dots, m$, we have that $1 - 1/y_i' > 0$. So

$$m - 1 - \sum_{i=1}^{m} \frac{1}{y_i'} = \left(1 - \frac{1}{y_1'} - \frac{1}{y_2'}\right) + \sum_{i=3}^{m} \left(1 - \frac{1}{y_i'}\right) > 0.$$

By (18) we have that $B_{m+1}(y_1, \dots, y_m, x_k) > 0$. This completes the proof. ∎

LEMMA 6. *If* $|M^{(3)}| \leq 3$, *then* $\beta_4(y_1, y_2, y_3, x_k) \neq 0$.

Proof. By Lemma 5, we need only to consider the case $|M^{(3)}| = 2$ and the case $|M^{(3)}| = 3$.

If $|M^{(3)}| = 2$, then let $M^{(3)} = \{G, x \cdot G\}$, where $x > 1$. Then at least one of the three elements (y_i, y_j), $1 \leq i < j \leq 3$, is $x \cdot G$. Since $(y_1, y_2, y_3) = G$, so, exactly, one of the three elements (y_i, y_j), $1 \leq i < j \leq 3$, is $x \cdot G$, and the other two elements are G. By (4) we have that

$$\begin{aligned}
\beta_4(y_1, y_2, y_3, x_k) &= \frac{1}{x_k} - \sum_{i=1}^{3} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{2}{G} - \frac{1}{G} \\
&= \frac{1}{x_k} - \sum_{i=1}^{3} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{1}{G} \\
&= \frac{1}{x_k} + \frac{1}{G}\left(-\sum_{i=1}^{3} \frac{1}{y_i'} + \frac{1}{x} + 1\right). \qquad (19)
\end{aligned}$$

Let $(y_a, y_b) = x \cdot G$ and $(y_a, y_c) = (y_b, y_c) = G$, where $1 \leq a, b, c \leq 3$, and a, b, and c are distinct. Since y_a, y_b are the greatest-type divisors and $(y_a', y_b') = x$,

$$-\frac{1}{y_a'} - \frac{1}{y_b'} + \frac{1}{x} \geq -\frac{1}{2x} - \frac{1}{3x} + \frac{1}{x} > 0.$$

Note that $-1/y_c' + 1 > 0$. Then

$$-\sum_{i=1}^{3} \frac{1}{y_i'} + \frac{1}{x} + 1 = \left(-\frac{1}{y_a'} - \frac{1}{y_b'} + \frac{1}{x}\right) + \left(-\frac{1}{y_c'} + 1\right) > 0.$$

By (19) we have that $\beta_4(y_1, y_2, y_3, x_k) > 0$.

If $|M^{(3)}| = 3$, then let $M^{(3)} = \{G, x \cdot G, y \cdot G\}$, where $1 < x < y$. Then there are two of the three elements (y_i, y_j), $1 \le i < j \le 3$, such that one is $x \cdot G$ and the other is $y \cdot G$. It follows from $(y_1, y_2, y_3) = G$ that $(x, y) = 1$. Furthermore, exactly one of the three elements (y_i, y_j), $1 \le i < j \le 3$, is $x \cdot G$, one is $y \cdot G$, and one is G. By (4) we have that

$$\begin{aligned} \beta_4(y_1, y_2, y_3, x_k) &= \frac{1}{x_k} - \sum_{i=1}^{3} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{1}{y \cdot G} + \frac{1}{G} - \frac{1}{G} \\ &= \frac{1}{x_k} - \sum_{i=1}^{3} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{1}{y \cdot G} \\ &= \frac{1}{x_k} + \frac{1}{G} \cdot \left(- \sum_{i=1}^{3} \frac{1}{y_i'} + \frac{1}{x} + \frac{1}{y} \right). \end{aligned} \tag{20}$$

Let $(y_d, y_e) = x \cdot G$, $(y_d, y_f) = y \cdot G$, where $1 \le d, e, f \le 3$ and d, e, f are distinct. Since y_d, y_e are the greatest-type divisors and $(y_d', y_e') = x$,

$$-\frac{1}{y_d'} - \frac{1}{y_e'} + \frac{1}{x} \ge -\frac{1}{2x} - \frac{1}{3x} + \frac{1}{x} > 0.$$

Since y_f is the greatest-type divisor and $(y_d', y_f') = y$, $y_f' > y$. Then $-1/y_f' + 1/y > 0$. Hence

$$-\sum_{i=1}^{3} \frac{1}{y_i'} + \frac{1}{x} + \frac{1}{y} = \left(-\frac{1}{y_d'} - \frac{1}{y_e'} + \frac{1}{x} \right) + \left(-\frac{1}{y_f'} + \frac{1}{y} \right) > 0.$$

By (20) we have $\beta_4(y_1, y_2, y_3, x_k) > 0$. The proof is complete. ∎

LEMMA 7. *If* $|M^{(4)}| \le 2$, *then* $\beta_5(y_1, y_2, y_3, y_4, x_k) \ne 0$.

Proof. By Lemma 5, we need only to consider the case $|M^{(4)}| = 2$.

Now let $|M^{(4)}| = 2$. Then $M^{(4)} = \{G, x \cdot G\}$ for some $x > 1$. So at least one of the six elements (y_i, y_j), $1 \le i < j \le 4$, is $x \cdot G$. Since $(y_1, y_2, y_3, y_4) = G$, at most three of the six elements (y_i, y_j), $1 \le i < j \le 4$, are $x \cdot G$, and for $1 \le i \le 4, 1 \le i_1 < i_2 < i_3 \le 4$ and $i_j \ne i$ $(j = 1, 2, 3)$, we have either $(y_i, y_{i_1}) \ne x \cdot G$, $(y_i, y_{i_2}) \ne x \cdot G$, or $(y_i, y_{i_3}) \ne x \cdot G$. Consider the following cases:

Case 1. If exactly one of the six elements (y_i, y_j), $1 \le i < j \le 4$, is $x \cdot G$, then the other five elements are G. Hence, the four elements

(y_i, y_j, y_l), $1 \le i < j < l \le 4$, are G. By (4) we have that

$$\begin{aligned}\beta_5(y_1, y_2, y_3, y_4, x_k) &= \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{5}{G} - \frac{4}{G} + \frac{1}{G} \\ &= \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{1}{x \cdot G} + \frac{2}{G} \\ &= \frac{1}{x_k} + \frac{1}{G} \cdot \left(- \sum_{i=1}^{4} \frac{1}{y_i'} + \frac{1}{x} + 2 \right). \end{aligned} \tag{21}$$

Since $2 \le y_1' < y_2' < y_3' < y_4'$, $-\sum_{i=1}^{4} 1/y_i' + 2 \ge 2 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} > 0$. By (21) we have that $\beta_5(y_1, y_2, y_3, y_4, x_k) > 0$.

Case 2. If at least two of the six elements (y_i, y_j), $1 \le i < j \le 4$, are $x \cdot G$, then there do not exist four distinct positive integers e, f, g, h, $1 \le e, f, g, h \le 4$, such that $(y_e, y_f) = (y_g, y_h) = x \cdot G$. Otherwise we have that $(y_1, y_2, y_3, y_4) = (y_e, y_f, y_g, y_h) = ((y_e, y_f), (y_g, y_h)) = x \cdot G$. It contradicts to $(y_1, y_2, y_3, y_4) = G$. So there are only the following four cases:

(i) $(y_1, y_2) = (y_1, y_3) = x \cdot G$. Then $(y_2, y_3) = x \cdot G$ and the other three elements (y_i, y_j) are G;

(ii) $(y_1, y_2) = (y_1, y_4) = x \cdot G$. Then $(y_2, y_4) = x \cdot G$ and the other three elements (y_i, y_j) are G;

(iii) $(y_1, y_3) = (y_1, y_4) = x \cdot G$. Then $(y_3, y_4) = x \cdot G$ and the other three elements (y_i, y_j) are G;

(iv) $(y_2, y_3) = (y_2, y_4) = x \cdot G$. Then $(y_3, y_4) = x \cdot G$ and the other three elements (y_i, y_j) are G.

Then it can be deduced that exactly three of the six elements (y_i, y_j), $1 \le i < j \le 4$, are $x \cdot G$, and the other three elements are G. Thus, exactly one of the four elements (y_i, y_j, y_l), $1 \le i < j < l \le 4$, is $x \cdot G$, and the other three elements are G. By (4) we have that

$$\begin{aligned}\beta_5(y_1, y_2, y_3, y_4, x_k) &= \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{3}{x \cdot G} + \frac{3}{G} - \frac{1}{x \cdot G} - \frac{3}{G} + \frac{1}{G} \\ &= \frac{1}{x_k} - \sum_{i=1}^{4} \frac{1}{y_i} + \frac{2}{x \cdot G} + \frac{1}{G} \\ &= \frac{1}{x_k} + \frac{1}{G} \cdot \left(- \sum_{i=1}^{4} \frac{1}{y_i'} + \frac{2}{x} + 1 \right). \end{aligned} \tag{22}$$

Let $(y_a, y_b) = (y_a, y_c) = (y_b, y_c) = x \cdot G$, where $1 \leq a, b, \ c \leq 4$, and a, b, c are distinct. Since y_a, y_b, y_c are the greatest-type divisors of x_k, $\{y_a', y_b', y_c'\}$ has one element not less than $2x$, one not less than $3x$, and one not less than $4x$. Thus

$$-\frac{1}{y_a'} - \frac{1}{y_b'} - \frac{1}{y_c'} + \frac{2}{x} \geq \frac{2}{x} - \frac{1}{2x} - \frac{1}{3x} - \frac{1}{4x} > 0.$$

Now let $\{y_d'\} = \{y_1', y_2', y_3', y_4'\} \setminus \{y_a', y_b', y_c'\}$. Then $-1/y_d' + 1 > 0$. Therefore,

$$-\sum_{i=1}^{4} \frac{1}{y_i'} + \frac{2}{x} + 1 = \left(-\frac{1}{y_a'} - \frac{1}{y_b'} - \frac{1}{y_c'} + \frac{2}{x}\right) + \left(-\frac{1}{y_d'} + 1\right) > 0.$$

By (22) we have that $\beta_5(y_1, y_2, y_3, y_4, x_k) > 0$. The proof is complete. ∎

III. THE MAIN THEOREM

In this section, we give the main result of this paper as follows.

THEOREM. *Let n be a positive integer.*

(i) *If $n \leq 7$, then the Bourque–Ligh's conjecture is true. Namely, for any* gcd-*closed set $S = \{x_1, \ldots, x_n\}$ of n distinct positive integers, the* LCM *matrix $[S]_n$ defined on S is nonsingular.*

(ii) *If $n \geq 8$, then there exists a* gcd-*closed set $S = \{x_1, \ldots, x_n\}$ of n distinct positive integers, such that $\alpha_n(x_1, \ldots, x_n) = 0$. Therefore the Bourque–Ligh conjecture is not true for $n \geq 8$.*

Proof. (i) Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set of n distinct positive integers. Without loss of generality, we may let $1 \leq x_1 < x_2 < \cdots < x_n$. For $1 \leq k \leq n$, let $R_k = \{y_1, \ldots, y_m\}$ be the set of the greatest-type divisors of x_k in S, where $y_1 < \cdots < y_m$. Then $R_1 = \phi$ and $R_k \neq \phi$ (i.e., $m \geq 1$) for $k \geq 2$. By Lemma 3 we have that

$$\alpha_k = \beta_{m+1}(y_1, \ldots, y_m, x_k). \tag{23}$$

For $m \geq 2$, let $M_r^{(m)} = \{(y_{i_1}, \ldots, y_{i_r}) | 1 \leq i_1 < \cdots < i_r \leq m\}$ $(2 \leq r \leq m)$. Let $M^{(m)} = \bigcup_{r=2}^{m} M_r^{(m)}$. Clearly, $R_k \subset \{x_1, \ldots, x_{k-1}\}$. It follows from S is gcd-closed that $M^{(m)} \subset \{x_1, \ldots, x_{k-1}\}$. Since $y_1, \ldots, y_m$ are the greatest-type divisors of x_k in S, $R_k \cap M^{(m)} = \phi$. So $m + |M^{(m)}| \leq k - 1$. Then for $m \geq 2$, we have that

$$1 \leq |M^{(m)}| \leq k - m - 1. \tag{24}$$

We claim that $\alpha_k \neq 0$ for $1 \leq k \leq 7$.

Let $k = 1$. Then $\alpha_1 = 1/x_1 \neq 0$.

Let $k = 2$ or 3. Then $m = 1$. It follows from (23) and Lemma 4 that $\alpha_k = \beta_2(y_1, x_k) \neq 0$.

Let $k = 4$. Suppose that $m \geq 3$, then by (24) we have that $k \geq m + 2 \geq 3 + 2 = 5$. It is a contradiction and so $m \leq 2$. It follows from (23) and Lemma 4 that $\alpha_4 = \beta_{m+1}(y_1, \ldots, y_m, x_4) \neq 0$.

Let $k = 5$. If $m \leq 2$, then it follows from (23) and Lemma 4 that $\alpha_5 = \beta_{m+1}(y_1, \ldots, y_m, x_5) \neq 0$. If $m \geq 3$, by (24) we have that $|M^{(m)}| \leq 5 - m - 1 \leq 1$. It can be deduced that $|M^{(m)}| = 1$ and $m = 3$. It follows from (23) and Lemma 5 that $\alpha_5 = \beta_4(y_1, y_2, y_3, x_5) \neq 0$.

Let $k = 6$. If $m \leq 2$, then it follows from (23) and Lemma 4 that $\alpha_6 = \beta_{m+1}(y_1, \ldots, y_m, x_6) \neq 0$. If $m \geq 3$, by (24) we have that $m \leq k - 2 = 4$. Then $m = 3$ or 4. If $m = 3$, by (24) we have that $|M^{(3)}| \leq 6 - 3 - 1 = 2$. It follows from (23) and Lemma 6 that $\alpha_6 = \beta_4(y_1, y_2, y_3, x_6) \neq 0$. If $m = 4$, by (24) we have that $|M^{(4)}| \leq 6 - 4 - 1 = 1$. So $|M^{(4)}| = 1$. It follows from (23) and Lemma 5 that $\alpha_6 = \beta_5(y_1, y_2, y_3, y_4, x_6) \neq 0$.

Let $k = 7$. If $m \leq 2$, then it follows from (23) and Lemma 4 that $\alpha_7 = \beta_{m+1}(y_1, \ldots, y_m, x_7) \neq 0$. If $m \geq 3$, by (24) we have that $m \leq k - 2 = 5$. Then $m = 3$, 4, or 5. If $m = 3$, by (24) we have that $|M^{(3)}| \leq 7 - 3 - 1 = 3$. It follows from (23) and Lemma 6 that $\alpha_7 = \beta_4(y_1, y_2, y_3, x_7) \neq 0$. If $m = 4$, by (24) we have that $|M^{(4)}| \leq 7 - 4 - 1 = 2$. Then it follows from (23) and Lemma 7 that $\alpha_7 = \beta_5(y_1, y_2, y_3, y_4, x_7) \neq 0$. If $m = 5$, by (24) we have that $|M^{(5)}| \leq 1$. So $|M^{(5)}| = 1$. Then it follows from (23) and Lemma 5 that $\alpha_7 = \beta_6(y_1, y_2, y_3, y_4, y_5, x_7) \neq 0$.

Therefore the claim is true.

Now let $n \leq 7$. Then for k, $1 \leq k \leq n$, it follows from the claim that $\alpha_k \neq 0$. So by (1) we have that $\det[S]_n \neq 0$. Hence the LCM matrix $[S]_n$ defined on S is nonsingular. Thus the Bourque–Ligh conjecture is true for $n \leq 7$.

(ii) Let $n \geq 8$ and let $a > 1$ be any integer. Now let

$$\begin{gathered} x_i = a^{i-1}, \qquad 1 \leq i \leq n - 7, \\ x_{n-6} = 2b, \qquad x_{n-5} = 3b, \qquad x_{n-4} = 5b, \qquad x_{n-3} = 36b, \\ x_{n-2} = 230b, \qquad x_{n-1} = 825b, \qquad x_n = 227700b, \end{gathered}$$

where $b = a^{n-8}$. Then $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set of n distinct positive integers. It is easy to see that $\{x_{n-3}, x_{n-2}, x_{n-1}\}$ is the set of the greatest-type divisors of $x_n = 227700b$ in S. Since

$$\begin{aligned}
&\beta_4(x_{n-3}, x_{n-2}, x_{n-1}, x_n) \\
&\quad = \frac{1}{b} \cdot \beta_4(36, 230, 825, 227700) \\
&\quad = \frac{1}{b} \cdot \left(\frac{1}{227700} - \frac{1}{36} - \frac{1}{230} - \frac{1}{825} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 \right) \\
&\quad = 0.
\end{aligned}$$

It follows from Lemma 3 that $\alpha_n = \beta_4(x_{n-3}, x_{n-2}, x_{n-1}, x_n) = 0$. By (1) we have that $\det[S]_n = 0$, so the LCM matrix $[S]_n$ defined on S is singular. Thus the Bourque–Ligh conjecture is not true for $n \geq 8$. The proof is complete. ∎

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